POWER VECTORS AND ZERNIKE CONVERSION

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THIS CHAPTER WILL INCLUDE A REVIEW OF:

- Power vectors
- Zernike to spectacle prescription conversion

POWER VECTORS

NATURE OF SPHERO-CYLINDRICAL POWER

If all refractive errors could be corrected by spherical lenses, our job as optometrists would be much easier.

The problem is that most eyes have astigmatism in addition to spherical errors. Things become even more complex if we consider higher-order aberrations, but for the sake of discussion, we'll ignore them for now. As has been done in traditional optometry, we'll assume now that the refractive errors of an eye consist of just a spherical and astigmatic component.

Consider the following spectacle Rx: −3.00/ –1.00 x 010

Figure 2-1 shows that the power varies depending on the angular meridian.

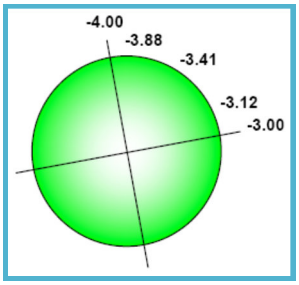


Figure 2-1:*Power in a sphero-cylinder lens varies according to meridian*

In other words, power changes as a function of meridian, and it changes according to the following formula:

$P(\theta) = S + C * [\sin 2(\theta - A)]$

(S is sphere power, C is cylinder power, A is axis (plus or minus cyl), P(θ) is the power in meridian θ.)

POWER VECTORS (CONT.)

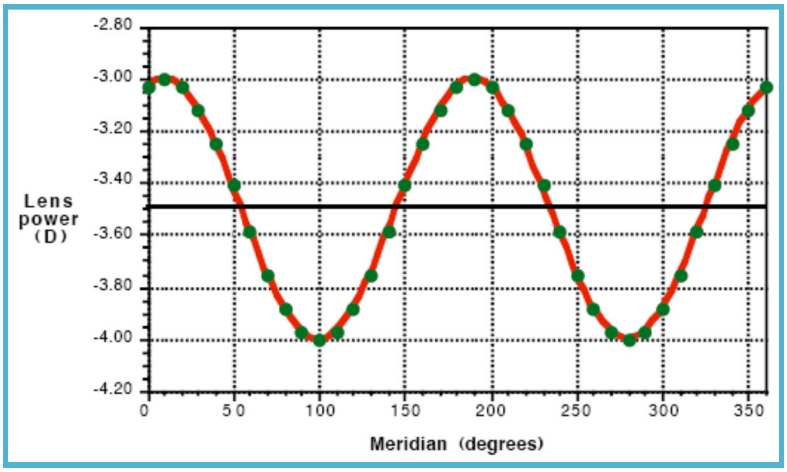


Figure 2-2: *Power profile of the sphero-cylindrical lens in Figure 2.1*

Figure 2-2 plots the power of this sphero-cylindrical lens as a function of meridian and shows that the power profile is a sinusoidal curve. If the lens had been a sphere, the power would be the same in all meridians, and it would have been represented by a horizontal line on the plot. The sinusoidal power change indicates astigmatism. The spherical equivalent power of this lens is –3.50, and is represented by the horizontal black line on Figure 2-2.

If we subtract the spherical equivalent, the entire curve will shift and be symmetric about zero, as shown in Figure 2-3. However, Figure 2-3 shows two curves. Notice that one is smaller, and is a sine function; the other is larger and is a cosine function. It turns out that the sum of these two curves will be another sinusoidal curve whose profile exactly matches the shape of the curve in Figure 2-2. If we then add the spherical equivalent power (−3.50 D) to the sum of the sine and cosine waves, the total will be identical to the power profile in Figure 2-2. This demonstrates that the power profile of a sphero-cylinder lens can be decomposed into three components:

- A spherical equivalent with some power (−3.50 in this case)
- A sine wave with some magnitude
- A cosine wave with some magnitude

If you have the correct power for each of these three functions, the sum of the three power functions will be exactly equal to the power profile of the sphero-cylinder lens.

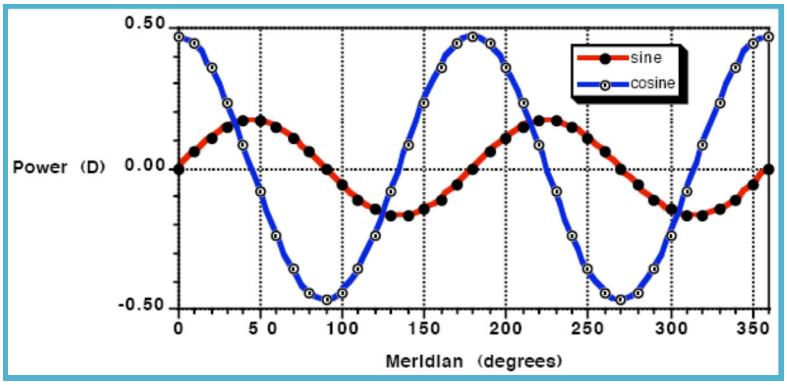


Figure 2-3:*The same astigmatic power profile can be expressed as the sum of a sine wave and a cosine wave, as shown here. By adjusting the magnitude of these two waves, we can ensure that their sum will match the sine wave profile (magnitude and axis) shown in on* ***Figure 2-2****. If we add the spherical equivalent power, the total power profile will equal the sphero-cyl profile plotted in* ***Figure 2-2***

Likewise, the power of any sphero-cylinder lens can be represented as the sum of a sine wave (scaled by an appropriate factor), a cosine wave (also scaled by the appropriate factor) and the spherical equivalent. Therefore, instead of writing the spectacle prescription in terms of sphere, cylinder, axis, it could also be written in terms of J45 (the sine-wave factor), M, which is the spherical equivalent, and J180 (the cosine wave factor). This will convert the clinical notation to what are known as power vectors.

POWER VECTORS (CONT.)

You can convert sphere (S), cylinder (C) and axis (A) to power vector notation using the following formulae:

$J45 = -(C/2)\sin(2A)$ …Equation 2.1

$M = S + C/2$ …Equation 2.2

$J180 = -(C/2)\cos(2A)$ …Equation 2.3

ADVANTAGES OF USING POWER VECTORS

Sometimes it is necessary to add, subtract, or compute the mean of several refractive measurements. This is clinically important, especially in contact lens practice. This is also important in vision research. Some specific examples would be:

- Adding an over-refraction to the contact lens power
- Subtracting corneal cylinder from spectacle cylinder to predict residual astigmatism in a gas permeable lens
- Computing the change in refractive error over time, when changes in cylinder may have occurred
- Computing the mean refractive error of a population

The clinical notation of sphere, cylinder and axis does not lend itself to these types of analyses. Obviously cylinders with different axes cannot be directly added together. You can, however add power vectors directly. This allows you to compute statistics such as the mean of several refractive errors.

CONVERTING FROM POWER VECTORS TO SPHERE, CYLINDER AND AXIS

$C = -2\sqrt{J_{45}^2 + J_{180}^2}$ …Equation 2.4

$S = M - \frac{C}{2}$ …Equation 2.5

$A = \frac{1}{2}\tan^{-1}\left(\frac{J_{45}}{J_{180}}\right)$ ...Equation 2.6

The computation of the axis (A) can be tricky in some cases, depending on the value of J45 and J180. The computation will yield an error if J180 is equal to zero. In some cases you'll get a negative value for the axis, which will have to be interpreted differently depending on the values of J180 and J45, whether they are positive, negative or equal to zero. To ensure that the final answer for the minus-cylinder axis is correct (positive value greater than 0 and less than or equal to 180), you should test the value of A and modify, if necessary, according to the following algorithm, which is also summarized in Table 2-1.

IF (J180=0, IF (J45<0, axis = 135, ELSE axis = 45), IF (J180<0, axis = A+90, ELSE IF (J45<=0, axis = A+180, ELSE axis = A)))

Table 2-1 Possible values for axis, based on value for J180 and J45. In order to correctly convert power vectors to the correct minus-cylinder axis, you must test the sign of the J180 and J45 (+ positive, - negative or zero), and modify the value of A computed in Equation 2.6 according to the "Axis" column below.

CASE	J_{180}	J_{45}	AXIS	VECTOR ON FIGURE 2-4	DESCRIPTION
1	+	0	180	Positive x axis	axis = 180
2	+	+	A	Quadrant 1	0<axis<45
3	0	+	45	Positive y axis	axis=45
4	−	+	A+90	Quadrant 2	45<axis<90
5	−	0	90	Negative x axis	axis=90
6	−	−	A+90	Quadrant 3	90<axis<135
7	0	−	135	Negative y axis	axis=135

POWER VECTORS (CONT.)

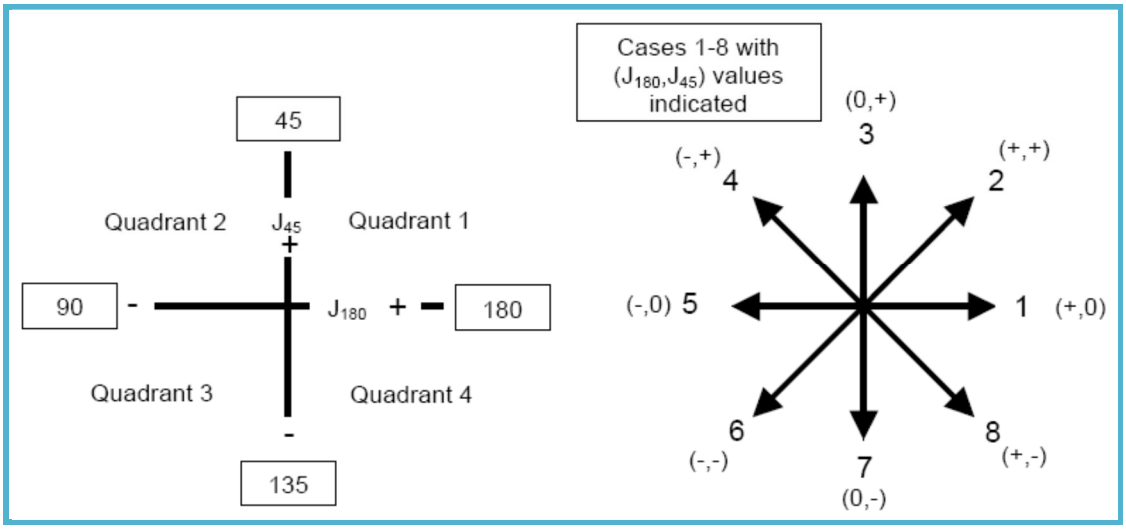


Figure 2-4: *Power vectors*

If the power vectors J_{180} and J_{45} are laid out as shown in the left figure, with J_{180} as the x-axis and J45 as the y-axis, and the full circle equaling 180 degrees, you can easily identify the correct axis for a vector falling into any of the 8 cases, shown on the right figure. Parentheses indicate the values (+, - or 0) for (J_{180}, J_{45}), respectively.

Example 1 – Toric soft contact lens

A patient is wearing a toric soft contact lens with a power equal to: -3.00 / –1.00 x 180. It rotates 15 degrees counter clockwise, so on the eye, the power becomes –3.00 / –1.00 x 15. The over-refraction is: +0.25 / –1.50 x 140. What is the correct sphero-cylinder prescription for this eye?

In this case you must add the over-refraction, a sphere-cylinder power, to the contact lens power, also a sphero-cylinder. To do this, you should

1. Convert both prescriptions to power vectors using Equations 2.1 - 2.3.
2. Add the corresponding terms of the power vectors
3. Convert the summated power vector back to sphere, cylinder and axis using Equations 2.4 - 2.6.

Table 2-2: *Data and solution for Example 1*

STEP 1. COMPUTE VECTOR	S	C	A	J_{45}	M	J_{180}
SCL on the eye	-3.00	-1.00	15	**0.250**	**-3.50**	**0.433**
Over-refraction	+0.25	-1.50	140	**-0.739**	**-0.500**	**0.130**

STEP 2. SUM EACH COLUMN				**-0.489**	**-4.000**	**0.563**
STEP 3. CONVERT BACK	**S**	**C**	**A**			
	–3.25	**-1.49**	**160**			

In the last step, you may get an axis of –20. In ophthalmic terms, this is the same as 160.

POWER VECTORS (CONT.)

Example 2 – Mean of two refractions

One student refracts a patient's right eye and gets –3.00 / –1.00 x 15. Another student refracts the same eye and gets –2.75 / -1.25 x 5. What is the mean of these two refractions? Do similar calculations as above, but instead of adding the two power vectors, compute the mean for each column.

Table 2-3: *Data and solution for Example 2*

STEP 1. COMPUTE VECTOR	S	C	A	J_{45}	M	J_{180}
Student 1	-3.00	-1.00	15	**0.250**	**-3.50**	**0.433**
Student 2	-2.75	-1.25	5	**0.109**	**-3.375**	**0.616**

STEP 2. FIND THE MEAN FOR EACH COLUMN				**0.179**	**-3.438**	**0.524**
STEP 3. CONVERT BACK	**S**	**C**	**A**			
	−2.88	**-1.11**	**9**			

Q. What would you get if you didn't use power vectors and took the mean of axis 5 and 15?

Example 3 – How much did a prescription change?

The refraction for a patient one year ago was: −3.00 / –1.00 x 15. Today the same eye is: –3.00 / –1.00 x 75. How much has the sphero-cylindrical prescription changed? Notice that both prescriptions have the same spherical equivalent, so if you ignore the axis, you might assume that very little has changed. Solve this by subtracting the first prescription, from the second, using power vectors.

Table 2-4: *Data and solution for Example 3*

STEP 1. COMPUTE VECTOR	S	C	A	J_{45}	M	J_{180}
Rx one year ago	−3.00	−1.00	015	**0.250**	**−3.50**	**0.433**
Rx today	−3.00	−1.00	075	**0.250**	**−3.50**	**−0.433**

STEP 2. SUBTRACT RX TODAY MINUS RX ONE YEAR AGO				**0.000**	**0.000**	**-0.866**
STEP 3. CONVERT BACK	**S**	**C**	**A**			
	+0.87	**−1.73**	**090**			

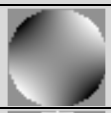
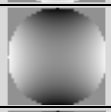
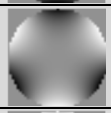
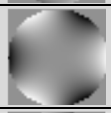
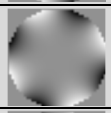
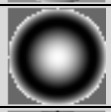
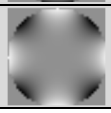
Although these computations may seem tedious to do by hand, they may be easily programmed into a spreadsheet such as Microsoft Excel. One such spreadsheet is available on the Vision Science II web page.

http://arapaho.nsuok.edu/~salmonto/vs2_lectures/PowerVector.xls

ZERNIKE TO SPECTACLE RX CONVERSION

Aberrometers measure lower and higher-order aberrations and report them using Zernike polynomials. Each polynomial represents a particular aberration. Table 2-5 lists the Zernike polynomials, that is, the aberrations in the second through fourth orders.

Table 2-5:*Zernike polynomials in the second through fourth orders*

ORDER	ZERNIKE MODE	NAME	EQUATION (POLAR)	TOPO MAP
2	Z_2^{-2}	Oblique astigmatism	$\sqrt{6}\rho^2\sin 2\theta$	
2	Z_2^{0}	Spherical defocus	$\sqrt{3}(2\rho^2-1)$	
2	Z_2^{2}	WTR/ATR* astigmatism	$\sqrt{6}\rho^2\cos 2\theta$	
3	Z_3^{-3}	Oblique trefoil	$\sqrt{8}\rho^3\sin 3\theta$	
3	Z_3^{-1}	Vertical coma	$\sqrt{8}(3\rho^3-2\rho)\sin\theta$	
3	Z_3^{1}	Horizontal coma	$\sqrt{8}(3\rho^3-2\rho)\cos\theta$	
3	Z_3^{3}	Horizontal trefoil	$\sqrt{8}\rho^3\cos 3\theta$	
4	Z_4^{-4}	Oblique quadrafoil	$\sqrt{10}\rho^4\sin 4\theta$	
4	Z_4^{-2}	Oblique 2nd order astigmatism	$\sqrt{10}(4\rho^4-3\rho^2)\sin 2\theta$	
4	Z_4^{0}	Spherical aberration	$\sqrt{5}(6\rho^4-6\rho^2+1)$	
4	Z_4^{2}	WTR/ATR* 2nd order astigmatism	$\sqrt{10}(4\rho^4-3\rho^2)\cos 2\theta$	
4	Z_4^{4}	Horizontal quadrafoil	$\sqrt{10}\rho^4\cos 4\theta$	

ZERNIKE TO SPECTACLE RX CONVERSION (CONT.)

Among the aberrations shown in Table 2-5, those in the second order are known as lower-order aberrations. Third-order aberrations and above are the higher-order aberrations.

The second-order aberrations are the refractive errors we routinely correct using spectacles, contact lenses and conventional LASIK. They include spherical defocus and astigmatism (which is divided up into two modes). The amount of each aberration present is expressed by a coefficient. The coefficient associated with a particular mode can be labeled by a variable, such as C, with a subscript and superscript corresponding to the mode. For example, coefficients C_2^{-2}, C_2^0 and C_2^2 respectively tell how much of the aberrations, Z_2^{-2}, Z_2^0, and Z_2^2 are present. Interestingly, the three second-order Zernike terms correspond with the three power vectors, J45, M and J180, so you can directly convert them to power vectors using Equations 2.7-2.9 (below), then, convert the power vectors to the equivalent sphere, cylinder and axis using Equations 2.4-2.6, as discussed above. Use the following specific steps:

1. Aberrometers usually report Zernike coefficients (C) in micrometers (1x10−6 m) and the pupil diameter in millimeters (1 x 10−3 m). Leave both in their original units
2. Divide the pupil diameter by 2 to get pupil radius (y in Equations 2.7-2.9). (Strictly speaking, you should convert everything to meters first, but it works out that, if you leave the coefficients in microns and pupil radius in mm, the final result will be in diopters.)
3. Using Equations 2.7-2.9, convert the coefficients C_2^{-2}, C_2^0 and C_2^2 into the respective power vectors J45, M and J180
4. Use Equations 2.4-2.6 to convert the power vectors J45, M and J180 to sphere, cylinder and axis, as before

$J45 = (-2\sqrt{6})C_2^{-2}/y^2$ …Equation 2.7

$M = (-4\sqrt{6})C_2^{0}/y^2$ …Equation 2.8

$J180 = (-2\sqrt{6})C_2^{2}/y^2$ …Equation 2.9

Example 4

Given the aberrometer data in Table 2-6, compute the equivalent sphere, cylinder and axis. Assume a pupil diameter of 6.0 mm.

Table 2-6: *Zernike polynomials in the second through fourth orders*

ABERROMETER DATA	C_2^{-2}	C_2^0	C_2^2
micrometers	−0.50	+1.50	−0.50
POWER VECTORS	**J45**	**M**	**J180**
diopters	0.27	−1.15	0.27
CLINICAL NOTATION	**SPHERE**	**CYLINDER**	**AXIS**
diopters	−0.77	−0.77	22.5°

If you compare the sign of the defocus coefficient (C_2^0) and the spherical power, you will notice that a positive wavefront error corresponds with a myopic error. These computations can be easily programmed into an Excel spreadsheet.