FOURIER TRANSFORMATION

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THIS CHAPTER WILL INCLUDE A REVIEW OF:

- Fourier analysis
- Characteristics of sine waves
- Two-dimensional functions
- Fourier transformations

FOURIER ANALYSIS

Joseph Fourier (1768-1830) was a mathematical physicist in Napoleon's army who was assigned to serve in Egypt. While there, he studied the propagation of heat in the ground and developed a new technique for analyzing mathematical functions. He discovered that no matter how complicated, any mathematical **function**, that is, any curve, can be broken down (analyzed) into basic components, which are sine waves. Just as a word can be broken down into components, letters, any mathematical function can be decomposed into basic elements – sine waves. With 26 letters we can write any word, and likewise, with just sine waves, we can make any function, that is, any curve.

Figure 12-1: *Joseph Fourier*

FOURIER ANALYSIS (CONT.)

A mathematical function is a string of numbers that varies in a systematic way. We say that the numbers vary as a function of something else, and often plot the data on x-y axes. In this case, we indicate how the value of a dependent variable (shown by the y-axis) changes according to (as a function of) the value of an independent variable (shown on the x-axis). We've already used these kinds of one-dimensional functions to study certain aspects of vision, such as luminous efficiency, dark adaptation, and spectral sensitivity of photoreceptors. The 1924 CIE V(λ) function is shown in Figure 12-2.

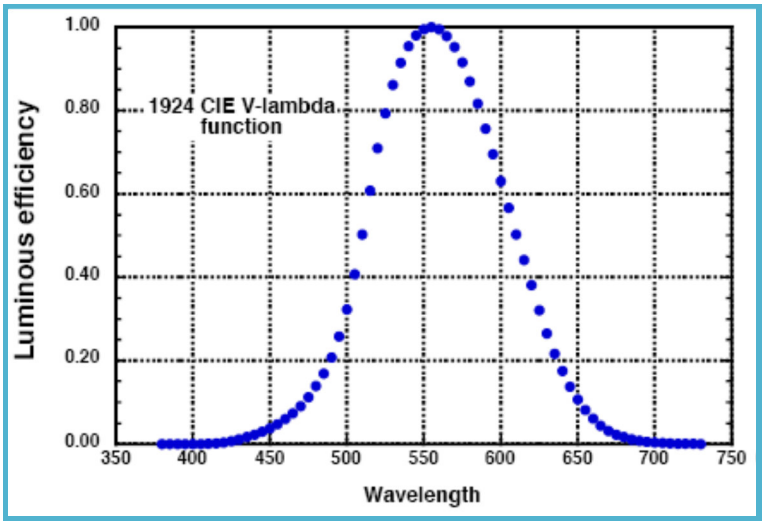

Figure 12-2: *A one-dimensional function*

Using Fourier analysis, we can break this (or any) function down into sub-components, all of which are sine waves, which will vary in terms of their frequencies and magnitudes.

Fourier analysis is used to analyze and study functions in many fields of science, engineering and business.

CHARACTERISTICS OF SINE WAVES

Figure 12-3 shows one example of a sine wave grating and Figure 12.4 shows its luminance profile, which is a sine wave.

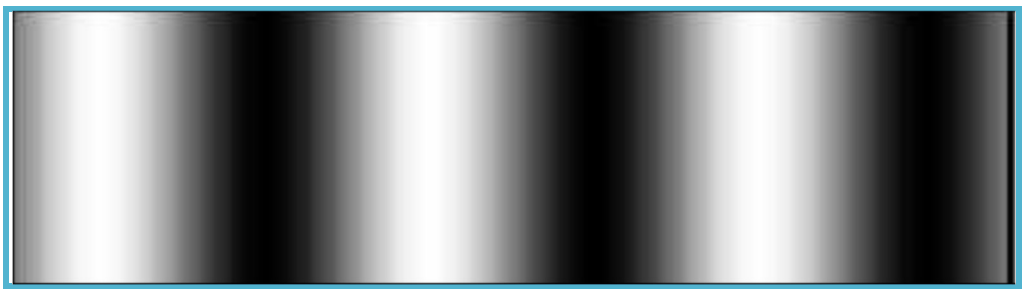

Figure 12-3: *A one-dimensional function*

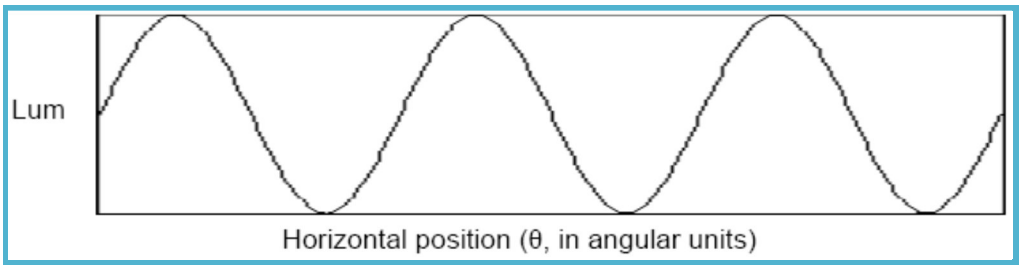

Figure 12-4: *Luminance profile of the sine wave grating in* ***Figure 12-3***

CHARACTERISTICS OF SINE WAVES (CONT.)

To understand the basic characteristics of sine wave gratings, let us consider a luminance profile such as the one above, which is a one-dimensional function. A sine wave grating, as well as its one-dimensional luminance profile, which is a sine wave, can be altered in the following ways:

- **Frequency** – how many cycles are contained within a defined span (period)
- **Amplitude** – how high are the peaks and how low are the valleys; this is related to **contrast**
- **Phase – is the waveform shifted to the right or left?**

The basic formula for a sine wave, for example the luminance (L) of the pattern above, as function of angular position (θ) is:

$$L = \sin(\theta)$$

In this case, you could plot the value of L along the y-axis and the value of θ would be along the x-axis. A more complete formula that allows us to modify the basic attributes of a sine wave is:

$$L = \Delta I * \sin(f * \theta - p) + Lm$$

Where the variables represent the following:

L = luminance value at one point on the sine wave
ΔI = difference between the peak and mean value (amplitude)
f = frequency in terms of number of cycles per degree
θ = horizontal angular position, in radians, or visual angle
p = phase shift of the entire sine wave
L_m = mean luminance

We will study how the different parameters affect the appearance of sine waves, and we will also look at examples of Fourier analysis.

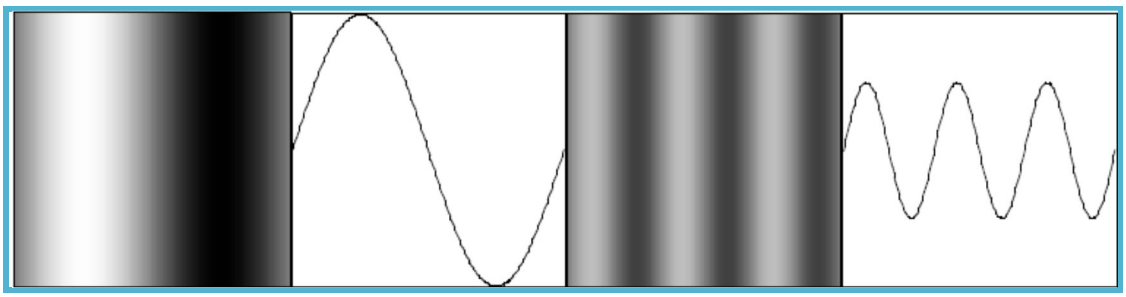

Figure 12-5: *Examples of vertically oriented sine wave gratings that differ in terms of contrast and spatial frequency*

TWO-DIMENSIONAL FUNCTIONS

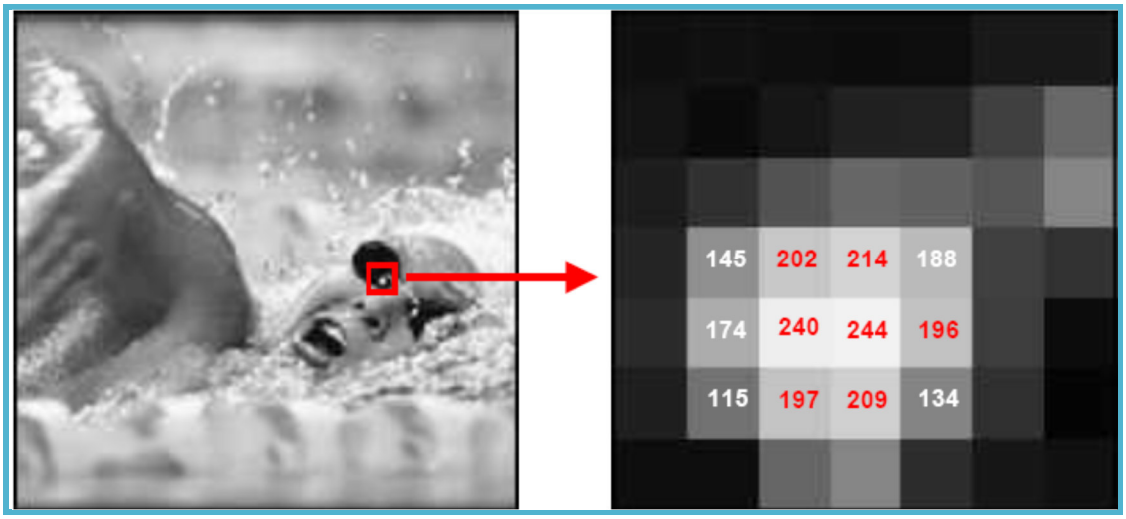

Figure 12-6: *Image of an Olympic swimmer (256x256 array, with 65,536 pixels): The magnified pixels (right) come from the white spot on the right swimming goggle. Numbers show the brightness values of some pixels. A value of 1 corresponds with black; 256, white; and everything in between is a gray level*

An **image** may be described as a **two-dimensional function** in which luminance values vary across two-dimensional space. The following data array shows the gray-scale values taken from the area surrounding the dot in Figure 12-6:

150	14140	183	175	14143
145	202	214	188	158
174	240	244	196	14141
115	197	209	134	158
75	173	192	132	87

The same techniques Joseph Fourier developed to work with one-dimensional functions can be applied to two-dimensional functions such as spatial images. When we perform Fourier analysis on a one-dimensional function, we break it down into its sine wave components, which are mathematical curves. When we Fourier analyze a two-dimensional image, we break it down into its sine wave grating components.

This allows us to study the way the visual system processes all images. Since it is a linear system, the manner in which it processes the components of images – sine wave gratings – is the same way it processes images that are made up of sine wave gratings; that is, all images.

Review questions:

Q. In what sense does Snellen visual acuity provide only a very limited assessment of vision?

A. . _____

Q. What is a function (one-dimensional)?

A. . _____

TWO-DIMENSIONAL FUNCTIONS (CONT.)

Q. In what sense is an image a 2-D function?

A. _____

Q. Why are we interested in breaking images down into basic building blocks such as sine wave gratings?

A. _____

FOURIER TRANSFORMATION

Fourier transformation is a process that computes the sine waves (the frequencies, magnitudes, phase shifts) contained in a mathematical function. In the case of an image (a two-dimensional function), it computes the sine wave gratings contained in the function. This process – Fourier transformation – takes an input data set and transforms it into the appropriate output data set, as illustrated in Figure 12-7.

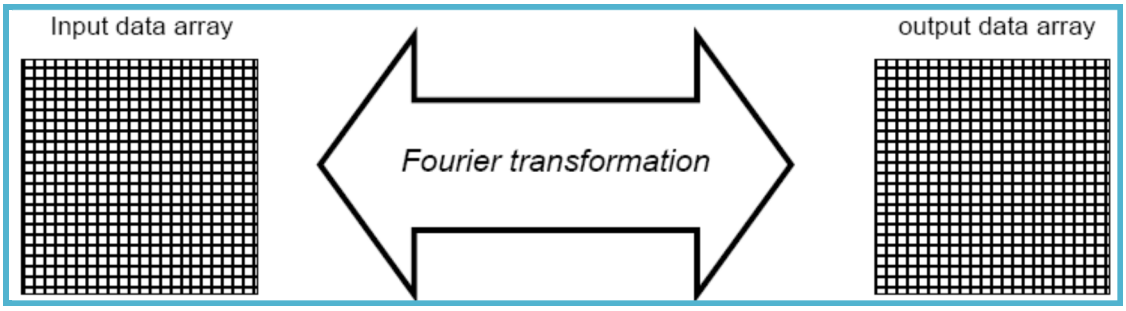

Figure 12-7: *Input and output arrays are linked by the process of Fourier transformation*

Some mathematical processes transform a number into another corresponding value, such as the reciprocal operation. Fourier transformation operates on an entire input data set [f(x,y)] and computes the appropriate corresponding output data set [F(u,v)] according to the following mathematical rule:

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\exp[-i2\pi(ux + vy)]dxdy$$

It is not necessary to understand how the rule works in order to understand what happens to the input and output data sets. In the case of an image (a two-dimensional distribution of luminance values across space) it transforms the image into another two-dimensional data array, which contains information about the sine wave gratings contained in the image (spatial frequencies, contrasts, orientations, phase shifts). The input and output data sets are related to each other by the process of Fourier transformation and are sometimes called **Fourier transform pairs**. In the case of images, the spatial image and its spatial frequency spectrum are Fourier transform pairs (Figure 12-8).

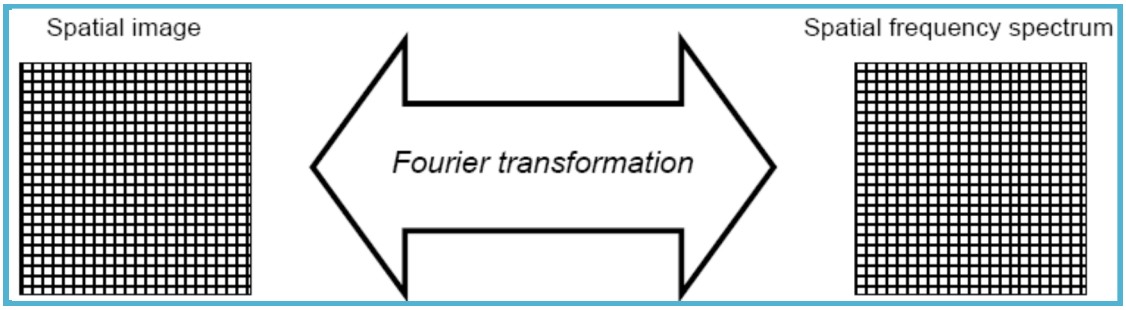

Figure 12-8: *A spatial image and its spatial frequency spectrum - Fourier transform pairs*

FOURIER TRANSFORMATION (CONT.)

One example of a Fourier transform pair (a spatial image and its spatial frequency spectrum) is show in Figure 12-9.

Figure 12-9: *A spatial image (left) and its spatial frequency spectrum (right)*

What do these mathematical processes have to do with vision? It turns out that optical systems, including the human eye, perform a Fourier transformation on light as it passes though the system. For example, the distribution of light in the exit pupil and the retinal image are Fourier transform pairs. In addition, scientists believe that the brain performs a Fourier transformation on the retinal image in the process of creating our sense of vision. That is, it takes the retinal image, breaks it down into the sine wave gratings that make up the image, sends the data describing the gratings back to the brain and reassembles the gratings into the image we perceive.
This is illustrated in Figure 12-10.

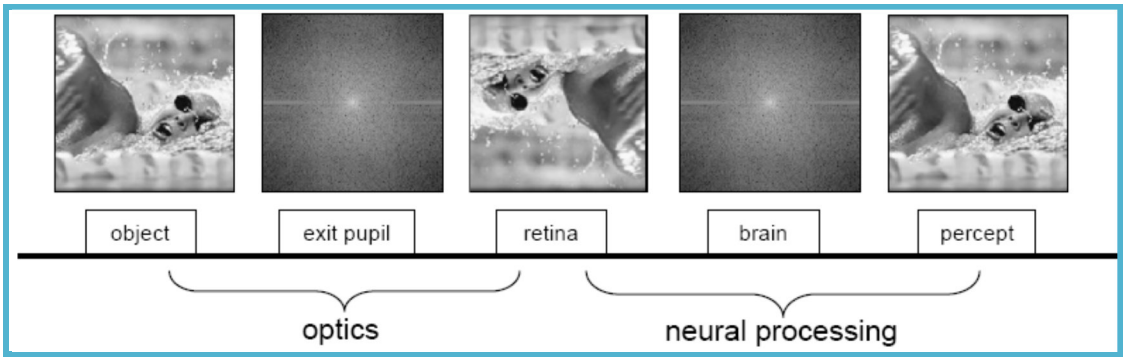

Figure 12-10: *Fourier transformation in the process of vision*